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Walker 4-manifolds with proper almost complex structures

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Abstract

It is shown that a Walker 4-manifold, endowed with a canonical neutral metric depending on three arbitrary functions, admits a specific almost complex structure (called *proper*) and an associated opposite almost complex structure. We study when these two almost complex structures are integrable and when the corresponding Kähler forms are symplectic. The conditions for the canonical neutral metric to be Kähler imply that the three arbitrary functions in the metric are all harmonic with respect to two coordinate variables, and we obtain a useful method of constructing indefinite Kähler 4-manifolds. Petean's example of a nonflat indefinite Kähler–Einstein 4-manifold is a special case of this construction.

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1. Introduction

By a Walker n-manifold, we mean a pseudo-Riemannian n-manifold which admits a field of parallel null r-planes, with $r \leq \frac{n}{2}$. It is known that an orientable Walker 4-manifold

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(M, g, D) (g: Walker metric, D: a field of parallel null 2-planes) admits an almost complex structure J and an opposite almost complex structure J' (cf. [4–7]). In the previous paper [7], a family of Walker 4-manifolds specified by a certain restriction on the metric g ($c = 0$) was studied, and Petean's nonflat indefinite Kähler–Einstein metric on a torus was obtained as an example of a Walker 4-manifold [9].

The purpose of the present note is to study certain almost complex structures J (called proper) on generic Walker 4-manifolds, and their associated opposite almost complex structures J'. We are interested in the integrability of J, J', and in the closure of the corresponding Kähler forms Ω, Ω', i.e., whether they are symplectic or not. There are, then, 16 possibilities according to whether J and J' are integrable or not and to whether Ω and Ω' are symplectic or not. In the restricted situation of [7], the proper almost complex structure studied in the present paper coincides with the almost complex structures defined in [7 (15)].

Our main result (Theorem 4) asserts that the conditions for a Walker 4-manifold to admit an indefinite Kähler structure imply that functions a, b and c which determine the metric g are all harmonic with respect to two coordinate variables. On the basis of this fact, we can easily construct numerous examples of indefinite Kähler 4-manifolds, including Petean's example [9] (see remark at the end of Section 3).

We also obtain Haze's example of a noncompact indefinite almost Kähler–Einstein 4-manifold which is not indefinite Kähler. This is an indefinite version of the example given by Nurowski and Przanowski [8].

Thus Walker 4-manifolds (M, g, D) display a large variety of indefinite geometry in four-dimension (cf. [1]).

2. A proper almost complex structure J and Kähler form Ω

2.1. Walker metric g

A Walker 4-manifold is a triple (M, g, D) consisting of a 4-manifold M, together with an indefinite metric g and a nonsingular field of two-dimensional planes D (or distribution) such that D is parallel and null with respect to g. From Walker's theorem [10, Theorem 1 and Case 1], there is a system of coordinates (x^1, x^2, x^3, x^4) with respect to which g takes the canonical form

$$g = [g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix}, \tag{1}$$

where a, b and c are functions of the coordinates (x^1, x^2, x^3, x^4). We see that g is of signature $(++--)$ (or neutral). The parallel null 2-plane D is spanned locally by $\{\partial_1, \partial_2\}$, where ∂_i are the abbreviated forms of $\frac{\partial}{\partial x^i}$, $(i = 1, \ldots, 4)$.

2.2. Proper almost complex structure J

We call a g-orthogonal almost complex structure J on a Walker 4-manifold M *proper* if J defines a standard generator of a positive $\frac{\pi}{2}$-rotation on D, i.e., explicitly

$$J\partial_1 = \partial_2, \qquad J\partial_2 = -\partial_1. \tag{2}$$

The following is a fundamental fact for the present issue.

Fact 1. *The canonical form (1) of g defines a unique proper almost complex structure J on a Walker 4-manifold M, namely the J defined by the following action on the coordinate basis*:

$$J\partial_1 = \partial_2, \qquad J\partial_2 = -\partial_1, \qquad J\partial_3 = -c\partial_1 + \tfrac{1}{2}(a-b)\partial_2 + \partial_4,$$
$$J\partial_4 = \tfrac{1}{2}(a-b)\partial_1 + c\partial_2 - \partial_3. \tag{3}$$

Proof. A proper almost complex structure J is characterized by the following three properties: (i) $J^2 = -1$, (ii) $g(JX, JY) = g(X, Y)$, and (iii) is standard on D as in (2).
It is straightforward to see that these three properties define J uniquely as in (3). □

If we write as $J\partial_i = \sum_{j=1}^{4} J_i^j \partial_j$, then from (3) we can read off the nonzero components J_i^j as follows:

$$J_1^2 = -J_2^1 = J_3^4 = -J_4^3 = 1, \qquad J_4^2 = -J_3^1 = c, \qquad J_3^2 = J_4^1 = \tfrac{1}{2}(a-b). \tag{4}$$

Remark. The proper almost complex structure J defined in (3) coincides with that defined in [7 (15)] in each of the cases (a) $c = 0$ and $a = b$, and case (b) $c = 0$ and $a = -b$. Note that in the former case (a), J is integrable (cf. [7, Proposition 4]).

2.3. Kähler form Ω

In terms of the metric g and the proper almost complex structure J, we can define a Kähler form $\Omega(X, Y) = g(JX, Y)$, whose explicit form is given by

$$\Omega = dx^1 \wedge dx^4 - dx^2 \wedge dx^3 + \tfrac{1}{2}(a+b)dx^3 \wedge dx^4. \tag{5}$$

Note that Ω is independent of the function c. We are interested in when Ω is symplectic, i.e., Ω is closed. (In what follows, we shall use the abbreviation $\partial p(x^1, x^2, x^3, x^4)/\partial x^i = \partial p/\partial x^i = p_i$, for any function p and $i = 1, \ldots, 4$.)

Theorem 2. *Ω is symplectic if and only if the sum $a + b$ is independent of x^1 and x^2. In fact, a and b satisfy the following PDEs*:

$$a_1 + b_1 = 0, \qquad a_2 + b_2 = 0. \tag{6}$$

Proof. These conditions follow directly from $d\Omega = \frac{1}{2}d(a+b) \wedge dx^3 \wedge dx^4 = 0$. □

Let q be a function of (x^1, x^2, x^3, x^4), and ϕ and ψ functions of (x^3, x^4), and put

$$a = a(x^1, x^2, x^3, x^4) = q(x^1, x^2, x^3, x^4) + \phi(x^3, x^4),$$
$$b = b(x^1, x^2, x^3, x^4) = -q(x^1, x^2, x^3, x^4) + \psi(x^3, x^4).$$

Then, a and b satisfy the PDEs in (6), and therefore the Kähler form becomes

$$\Omega = dx^1 \wedge dx^4 - dx^2 \wedge dx^3 + \tfrac{1}{2}(\phi(x^3, x^4) + \psi(x^3, x^4))dx^3 \wedge dx^4, \tag{7}$$

which is clearly closed.

2.4. J-integrability

The proper almost complex structure J in (3) is integrable if and only if the torsion of J (Nijenhuis tensor), with components

$$N^i_{jk} = 2\sum_{h=1}^{4}\left(J^h_j \frac{\partial J^i_k}{\partial x^h} - J^h_k \frac{\partial J^i_j}{\partial x^h} - J^i_h \frac{\partial J^h_k}{\partial x^j} + J^i_h \frac{\partial J^h_j}{\partial x^k}\right) \tag{8}$$

vanishes (cf. [3, p. 124]), where J_i^j are given by (4). From explicit calculations, we find the following J-integrability condition.

Theorem 3. *The proper almost complex structure J is integrable if and only if the following PDEs hold*:

$$a_1 - b_1 - 2c_2 = 0, \qquad a_2 - b_2 + 2c_1 = 0. \tag{9}$$

From this theorem, we immediately see that if $a = b$ and $c = 0$, then J is integrable (cf. remark below Fact 1 and [7, Proposition 4]).

2.5. Indefinite Kähler structure

As the main result of the present paper, we have the Kähler condition as follows.

Theorem 4. *The triple* (g, J, Ω) *is Kähler if and only if the following PDEs hold*:

$$a_1 = -b_1 = c_2, \qquad a_2 = -b_2 = -c_1. \tag{10}$$

Moreover, if the triple (g, J, Ω) *is Kähler, then the functions a, b and c are all harmonic with respect to the first two arguments* (x^1, x^2). *That is,*

$$a_{11} + a_{22} = 0, \qquad b_{11} + b_{22} = 0, \qquad c_{11} + c_{22} = 0. \tag{11}$$

Proof. The combination of PDEs in (6) and (9) gives the desired conditions (10). From these equations, we have $a_{11} = (a_1)_1 = (c_2)_1 = c_{12} = (c_1)_2 = -(a_2)_2 = -a_{22}$, and hence $a_{11} + a_{22} = 0$. Similarly, we can see that $b_{11} + b_{22} = 0$ and $c_{11} + c_{22} = 0$. □

3. Construction of indefinite Kähler 4-maniflolds

Theorem 4 provides a useful method of producing examples of indefinite Kähler 4-maniflolds, which we now explain.

We begin with a harmonic function $h(x, y)$ of two variables (x, y). That is, h is a solution to the following Laplace equation:

$$(\partial_{xx} + \partial_{yy})h(x, y) = 0. \tag{12}$$

Many harmonic functions $h(x, y)$ of two variables are known, e.g., as follows:

$$\begin{aligned} &x^2 - y^2, \quad 2x(1-y), \quad (x-y)(x^2+4xy+y^2), \quad x^3 - 3xy^2, \\ &3x^2y - y^3, \quad \cos x \sinh y, \quad e^x \cos y, \quad e^x(x \cos y - y \sin y), \\ &\log(x^2+y^2), \text{ etc.} \end{aligned} \tag{13}$$

We shall construct an indefinite Kähler 4-manifold, starting from a harmonic function, e.g. $h(x, y) = \cos x \sinh y$. First put $a = h(x^1, x^2) + \psi(x^3, x^4)$, i.e., as follows:

$$a = a(x^1, x^2, x^3, x^4) = \cos x^1 \sinh x^2 + \psi(x^3, x^4), \tag{14}$$

where ψ is an arbitrary smooth function of (x^3, x^4). Then, a is also harmonic with respect to (x^1, x^2), and we have

$$a_1 = -\sin x^1 \sinh x^2, \qquad a_2 = \cos x^1 \cosh x^2. \tag{15}$$

From (10), we have PDEs for b to satisfy as

$$b_1 = -a_1 = \sin x^1 \sinh x^2, \qquad b_2 = -a_2 = -\cos x^1 \cosh x^2, \tag{16}$$

and similarly PDEs for c to satisfy as

$$c_1 = -a_2 = -\cos x^1 \cosh x^2, \qquad c_2 = a_1 = -\sin x^1 \sinh x^2. \tag{17}$$

These PDEs are easily solved, and we have solutions

$$b = b(x^1, x^2, x^3, x^4) = -\cos x^1 \sinh x^2 + \lambda(x^3, x^4), \tag{18}$$

$$c = c(x^1, x^2, x^3, x^4) = \sin x^1 \cosh x^2 + \mu(x^3, x^4), \tag{19}$$

where $\lambda(x^3, x^4)$, $\mu(x^3, x^4)$ are arbitrary smooth functions of (x^3, x^4). Thus the indefinite Kähler metric takes the form

$$g = [g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \cos x^1 \sinh x^2 + \psi(x^3, x^4) & \sin x^1 \cosh x^2 + \mu(x^3, x^4) \\ 0 & 1 & \sin x^1 \cosh x^2 + \mu(x^3, x^4) & -\cos x^1 \sinh x^2 + \lambda(x^3, x^4) \end{bmatrix}. \tag{20}$$

Remarks

(i) We must consider the integrability conditions of the PDEs for b and c in (10) (or explicitly (16) and (17)). Suppose that $b_1 = f$ and $b_2 = g$ are the given PDEs for b for known functions f and g. It is well known that $f_2 = g_1$ is the integrability condition. In our case, as in (10), we see that $f = -a_1$ and $g = -a_2$, and therefore the integrability condition for b is always satisfied as $f_2 = -a_{12} = g_1$. Similarly, the system for c is also integrable. Thus, for any harmonic function a, there always exists solutions b and c, whence our procedure for constructing indefinite Kähler structures on Walker 4-manifolds is justified.

(ii) Petean's nonflat indefinite Kähler–Einstein metric [9] can be constructed in this way as a very special case. Assume first that $h = 0$. Then, we have that $a = \psi(x^3, x^4)$, $b = \lambda(x^3, x^4)$, and $c = \mu(x^3, x^4)$. If we further assume that $c = \mu(x^3, x^4) = 0$, and that $\psi(x^3, x^4) = \lambda(x^3, x^4)$, i.e., $a = b = \psi(x^3, x^4)$, then the metric becomes

$$g = [g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \psi(x^3, x^4) & 0 \\ 0 & 1 & 0 & \psi(x^3, x^4) \end{bmatrix}, \tag{21}$$

which is precisely Petean's example. Note that this is an example of the case $c = 0$ and $a = b$ (cf. remark below Fact 1).

4. Opposite almost complex structure J' and opposite Kähler form Ω'

4.1. Opposite almost complex structure

It is known that an oriented 4-manifold with a field of 2-planes, or equivalently endowed with a neutral indefinite metric, admits a pair of almost complex structure J and an opposite almost complex structure J', which satisfy the following properties (cf. [4–6]):

(i) $J^2 = J'^2 = -1$;
(ii) $g(JX, JY) = g(J'X, J'Y) = g(X, Y)$;
(iii) $JJ' = J'J$;
(iv) the preferred orientation of J coincides with that of M;
(v) the preferred orientation of J' is opposite to that of M.

Remark. Let $V = (\mathbb{R}^4, <, >)$ be a four-dimensional vector space with a quadratic form $<, >$ of neutral signature. Suppose that a complex structures I acting on V which keep the quadratic form $<, >$ invariant. All such complex structures can be obtained from some fixed complex structure, say I_0, by the action of $SO(2, 2)$, as $I = AI_0A^{-1}$, $A \in SO_0(2, 2)$. The isotropy subgroup of $SO(2, 2)$ at I_0 is the unitary group $U(1, 1)$ if the preferred orientation of I coincides with that of V. Similarly, we denote by $U'(1, 1)$ the isotropy subgroup of $SO(2, 2)$ which keep some fixed opposite complex structure I'_0 invariant. Then, all orthogonal complex structures can be identified with the quotient space $\{I\} \sim SO(2, 2)/U(1, 1)$. Similarly, all orthogonal opposite complex structures can be identified with the quotient space $\{I'\} \sim SO(2, 2)/U'(1, 1)$. At present, it is important to recognize a fact that $\dim SO(2, 2)/U(1, 1) = \dim SO(2, 2)/U'(1, 1) = 2$.

Since $\dim\{I'\} = 2$, the opposite almost complex structure J' associated with the proper almost complex structure J cannot be determined uniquely.

Proposition 5. *For a Walker 4-manifold (M, g, D), with the proper almost complex structure J, the g-orthogonal opposite almost complex structure J' takes the form*

$$J'\partial_1 = -\theta_2\partial_1 - \frac{\theta_1}{2}b\partial_2 + \theta_1\partial_4, \qquad J'\partial_2 = \frac{\theta_1}{2}a\partial_1 + (\theta_1 c - \theta_2)\partial_2 - \theta_1\partial_3,$$

$$J'\partial_3 = \left(\frac{\theta_1}{2}c - \theta_2\right)a\partial_1 + \left(\frac{1}{\theta_1} + \theta_1 c^2 - \frac{\theta_1}{4}ab - 2\theta_2 c + \frac{(\theta_2)^2}{\theta_1}\right)\partial_2 - (\theta_1 c - \theta_2)\partial_3 + \frac{\theta_1}{2}a\partial_4,$$

$$J'\partial_4 = -\left(\frac{1}{\theta_1} - \frac{\theta_1}{4}ab + \frac{(\theta_2)^2}{\theta_1}\right)\partial_1 + \left(\frac{\theta_1}{2}c - \theta_2\right)b\partial_2 - \frac{\theta_1}{2}b\partial_3 + \theta_2\partial_4,$$

where $\theta_1(\neq 0)$ and θ_2 are two parameters.

It may be interesting and significant to analyze if such a generic J' is integrable or not, and if the opposite Kähler form $\Omega'(X, Y) = g(J'X, Y)$ is symplectic or not. In the present paper, however, we shall focus our attention to one of explicit forms of J', obtained by fixing two parameters as $\theta_1 = 1$ and $\theta_2 = 0$ (only for simplicity), as follows:

$$J'\partial_1 = -\tfrac{1}{2}b\partial_2 + \partial_4, \qquad J'\partial_2 = \tfrac{1}{2}a\partial_1 + c\partial_2 - \partial_3,$$
$$J'\partial_3 = \tfrac{1}{2}ac\partial_1 + (1 - \tfrac{1}{4}ab + c^2)\partial_2 - c\partial_3 + \tfrac{1}{2}a\partial_4,$$
$$J'\partial_4 = -(1 - \tfrac{1}{4}ab)\partial_1 + \tfrac{1}{2}bc\partial_2 - \tfrac{1}{2}b\partial_3. \tag{22}$$

If we write as $J'\partial_i = \sum_{j=1}^{4} J'^{j}_{i}\partial_j$, then from (22) we can read off the nonzero components J'^{j}_{i} as follows:

$$J'^2_1 = -\tfrac{1}{2}b, \qquad J'^4_1 = 1, \qquad J'^1_2 = \tfrac{1}{2}a, \qquad J'^2_2 = c, \qquad J'^3_2 = -1,$$
$$J'^1_3 = \tfrac{1}{2}ac, \qquad J'^2_3 = 1 - \tfrac{1}{4}ab + c^2, \qquad J'^3_3 = -c, \qquad J'^4_3 = \tfrac{1}{2}a,$$
$$J'^1_4 = -1 + \tfrac{1}{4}ab, \qquad J'^2_4 = \tfrac{1}{2}bc, \qquad J'^3_4 = -\tfrac{1}{2}b. \tag{23}$$

Our analysis on J' in the present note is concerned only with J' defined just above. Therefore, we must take care that the results obtained in what follows are not concerned with the generic J'.

4.2. *Opposite Kähler form Ω'*

In terms of the metric g and the opposite almost complex structure J' in (22), we can define an opposite Kähler form $\Omega'(X, Y) = g(J'X, Y)$, whose explicit form is given by

$$\Omega' = dx^1 \wedge dx^2 + c\,dx^1 \wedge dx^3 + \tfrac{1}{2}b\,dx^1 \wedge dx^4$$

$$+(1+\tfrac{1}{4}ab)dx^3\wedge dx^4-\tfrac{1}{2}adx^2\wedge dx^3. \tag{24}$$

For the conditions for $\Omega'(X,Y)$ to be symplectic, we have the following theorem.

Theorem 6. *Ω' is symplectic ($d\Omega'=0$) if and only if the following PDEs hold*:

$$b_2=0,\qquad a_1+2c_2=0,\qquad ba_2-2a_4=0,\qquad ba_1+ab_1-2b_3+4c_4=0. \tag{25}$$

Proof. These PDEs can be obtained from the following differential of Ω':

$$d\Omega'=\tfrac{1}{4}((ab)_2-2a_4)dx^2\wedge dx^3\wedge dx^4+\tfrac{1}{4}((ab)_1-2b_3+4c_4)dx^1\wedge dx^3\wedge dx^4$$
$$-\tfrac{1}{2}b_2dx^1\wedge dx^2\wedge dx^4-\tfrac{1}{2}(a_1+2c_2)dx^1\wedge dx^2\wedge dx^3.\quad\square$$

4.3. J'-integrability

The opposite almost complex structure J' is integrable if the analogue of the partial derivatives (8) for ${J'}_i^j$ in (23) vanish. From some calculation, we have explicitly the following theorem.

Theorem 7. *The proper opposite almost complex structure J' is integrable if and only if the following PDEs hold*:

$$a_1=0,\qquad b_2+2c_1=0,\qquad ba_2-2a_4=0,$$
$$ab_1-2b_3-4cc_1-2bc_2+4c_4=0. \tag{26}$$

4.4. Opposite Kähler structure

The triple (g,J',Ω') is called an opposite Kähler structure if Ω' is symplectic and if J' is integrable.

Theorem 8. *The triple (g,J',Ω') is opposite Kähler if and only if the following PDEs hold*:

$$a_1=b_2=c_1=c_2=0,\qquad a_4=\tfrac{1}{2}ba_2,\qquad c_4=-\tfrac{1}{4}ab_1+\tfrac{1}{2}b_3. \tag{27}$$

Proof. The combination of (25) and (26) gives the desired PDEs as the condition to be opposite Kähler. $\square$

5. Sixteen classes of Walker 4-manifolds with respect to (g, J, Ω) and (g, J', Ω')

We now define various subfamilies in the set of all Walker 4-manifolds: $\mathcal{W}=\{M=(M,g,J,J',\Omega,\Omega')\}$.

$$\begin{tikzcd} & \mathcal{W} \arrow[dl] \arrow[dr] & \\ \mathcal{AK} \arrow[dr] & & \mathcal{H} \arrow[dl] \\ & \mathcal{K} & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{W} \arrow[dl] \arrow[dr] & \\ \mathcal{AK}' \arrow[dr] & & \mathcal{H}' \arrow[dl] \\ & \mathcal{K}' & \end{tikzcd}$$

$$(g, J, \Omega) \qquad\qquad (g, J', \Omega')$$

Plate 1.

- $\mathcal{AK} = \{M = (M, g, J, J', \Omega, \Omega')|d\Omega = 0\}$:
  Walker 4-manifolds with indefinite almost Kähler structure ($\rightarrow$ Theorem 2).
- $\mathcal{H} = \{M = (M, g, J, J', \Omega, \Omega')|J$ is integrable$\}$:
  Walker 4-manifolds with indefinite Hermitian structure ($\rightarrow$ Theorem 3).
- $\mathcal{K} = \{M = (M, g, J, J', \Omega, \Omega')|d\Omega = 0,\ J$ is integrable$\}$:
  Walker 4-manifolds with indefinite Kähler structure ($\rightarrow$ Theorem 4).
- $\mathcal{AK}' = \{M = (M, g, J, J', \Omega, \Omega')|d\Omega' = 0\}$:
  Walker 4-manifolds with indefinite opposite almost Kähler structure ($\rightarrow$ Theorem 6).
- $\mathcal{H}' = \{M = (M, g, J, J', \Omega, \Omega')|J'$ is integrable $\}$:
  Walker 4-manifolds with indefinite opposite Hermitian structure ($\rightarrow$ Theorem 7).
- $\mathcal{K}' = \{M = (M, g, J, J', \Omega, \Omega')|d\Omega' = 0,\ J'$ is integrable $\}$:
  Walker 4-manifolds with indefinite opposite Kähler structure ($\rightarrow$ Theorem 8).

We must note that $\mathcal{K} = \mathcal{AK} \cap \mathcal{H}$ and $\mathcal{K}' = \mathcal{AK}' \cap \mathcal{H}'$. See Plate 1, where arrows indicate from coarse to fine.

From these two kinds of subfamilies, we can further classify the Walker 4-manifolds into 16 classes as shown in Table 1.

Theorem 9. *The conditions for a Walker 4-manifold M to be in one of the following five subfamilies*:

$$\mathcal{K} \cap \mathcal{AK}', \qquad \mathcal{K} \cap \mathcal{H}', \qquad \mathcal{AK} \cap \mathcal{K}', \qquad \mathcal{H} \cap \mathcal{K}', \qquad \mathcal{K} \cap \mathcal{K}'$$

coincide with each other. In fact, such a condition is given explicitly as follows:

$$a_1 = a_2 = a_4 = b_1 = b_2 = c_1 = c_2 = 0, \qquad b_3 = 2c_4. \tag{28}$$

Table 1

$\mathcal{W}$	$\mathcal{AK}$	$\mathcal{H}$	$\mathcal{K}$
$\mathcal{AK}'$	$\mathcal{AK} \cap \mathcal{AK}'$	$\mathcal{H} \cap \mathcal{AK}'$	$\mathcal{K} \cap \mathcal{AK}'$
$\mathcal{H}'$	$\mathcal{AK} \cap \mathcal{H}'$	$\mathcal{H} \cap \mathcal{H}'$	$\mathcal{K} \cap \mathcal{H}'$
$\mathcal{K}'$	$\mathcal{AK} \cap \mathcal{K}'$	$\mathcal{H} \cap \mathcal{K}'$	$\mathcal{K} \cap \mathcal{K}'$

Table 2

$\mathcal{W}$	$\mathcal{AK}$	$\mathcal{H}$	$\mathcal{K}$
$\mathcal{AK}'$	$\mathcal{AK} \cap \mathcal{AK}'$	$\mathcal{H} \cap \mathcal{AK}'$	$\mathcal{K} \cap \mathcal{AK}'$
$\mathcal{H}'$	$\mathcal{AK} \cap \mathcal{H}'$	$\mathcal{H} \cap \mathcal{H}'$	$\mathcal{K} \cap \mathcal{H}'$
$\mathcal{K}'$	$\mathcal{AK} \cap \mathcal{K}'$	$\mathcal{H} \cap \mathcal{K}'$	$\mathcal{K} \cap \mathcal{K}'$

$$\mathcal{W}$$

$$\downarrow$$

$$\mathcal{AK} \qquad \mathcal{H} \qquad \mathcal{H}' \qquad \mathcal{AK}'$$

$$\downarrow$$

$$\mathcal{K}\,(=\mathcal{AK} \cap \mathcal{H}) \qquad \mathcal{AK} \cap \mathcal{H}' \qquad \mathcal{AK} \cap \mathcal{AK}' \qquad \mathcal{H} \cap \mathcal{H}' \qquad \mathcal{H} \cap \mathcal{AK}' \qquad \mathcal{K}'\,(=\mathcal{H}' \cap \mathcal{AK}')$$

$$\downarrow$$

$$\mathcal{K} \cap \mathcal{AK}' = \mathcal{K} \cap \mathcal{H}' = \mathcal{K} \cap \mathcal{K}' = \mathcal{H} \cap \mathcal{K}' = \mathcal{AK} \cap \mathcal{K}'$$

Plate 2.

Proof. Assume first that a Walker 4-manifold M is an indefinite Kähler 4-manifold: $M \in \mathcal{K}$, i.e., M satisfies the PDFs in (10). If M admits further an opposite symplectic form, i.e., $M \in \mathcal{AK}'$, then M must satisfy the PDEs in (25). Then, we see that for $M \in \mathcal{K} \cap \mathcal{AK}'$, these two kinds of conditions become the desired PDEs as in (28). For the other four subfamilies, such conditions are coincide with each other as in (28). □

These five subfamilies are indicated in the gray boxes in Table 2.

This theorem implies that if an indefinite Kähler 4-manifold ($M \in \mathcal{K}$) admits further, e.g., an integrable proper J' ($M \in \mathcal{H}'$), then the manifold ($M \in \mathcal{K} \cap \mathcal{H}'$) must be double Kähler, i.e., $M \in \mathcal{K} \cap \mathcal{K}'$.

From Theorem 9, it turns out that the Walker 4-manifolds are classified into essentially 12 subfamilies with respect to J, J', Ω, and Ω' (cf. Table 2 and Plate 2).

6. Curvatures characterized by J, J', Ω and Ω'

We have seen in the preceding sections the conditions for J and J' to be integrable, and those for Ω and Ω' to be closed. We can expect that the conditions for a Walker 4-manifold to be in some of the 16 subfamilies will restrict the curvature tensors to a certain extent. From such a point of view, we have some results as follows. Note that the curvature tensor R_{ijkl}, the Ricci tensor r_{ij}, the scalar curvature S, and the Einstein tensor G_{ij} are given in Appendices A–D.

Theorem 10. *If a Walker 4-manifold M is either opposite almost Kähler ($M \in \mathcal{AK}'$) or opposite Hermitian ($M \in \mathcal{H}'$), then M is scalar flat.*

Proof. Suppose that M is opposite almost Kähler. Then from the first two equations $b_2 = 0$ and $a_1 + 2c_2 = 0$ in (25) (Theorem 6), we see that $S = a_{11} + 2c_{12} + b_{22} = (a_1 + 2c_2)_1 = 0$.

Next, suppose that M is opposite Hermitian. Then from the first two equations $a_1 = 0$ and $b_2 + 2c_1 = 0$ in (26) (Theorem 7), we see that $S = a_{11} + 2c_{12} + b_{22} = (2c_1 + b_2)_2 = 0$. □

Theorem 11. *If a Walker 4-manifold M is in the subfamily: $\mathcal{K} \cap \mathcal{AK}' = \mathcal{K} \cap \mathcal{H}' = \mathcal{AK} \cap \mathcal{K}' = \mathcal{H} \cap \mathcal{K}' = \mathcal{K} \cap \mathcal{K}'$, then M is flat.*

Proof. If M is in the five subfamilies considered in Theorem 9, then the functions a, b and c satisfy the same conditions as in (28). Under such conditions, it is easy to see that the components R_{ijkl} of curvature tensor in Appendix A all vanish. □

7. Examples of indefinite Ricci flat almost-Kähler non-Kähler 4-manifolds

We show by construction an example, due to Haze [2], of noncompact indefinite Ricci flat almost-Kähler non-Kähler 4-manifolds. This is an indefinite version of the example given by Nurowski and Przanowski [8]. Consider the metric

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & 0 & -a \end{bmatrix}. \tag{29}$$

That is, the metric is defined by putting $b = -a$, $c = 0$ in the generic canonical form (1). From (5), we have that $\Omega = dx^1 \wedge dx^4 - dx^2 \wedge dx^3$, and hence is symplectic. In this case, we see from Appendix D that the Einstein condition ($G_{ij} = 0$) consists of the following PDEs:

$$\begin{aligned} &a_{12} = 0, \qquad aa_{11} - 2a_{24} - (a_2)^2 = 0, \qquad aa_{14} - a_{23} + a_1 a_2 = 0, \\ &aa_{11} - 2a_{13} + (a_1)^2 = 0. \end{aligned} \tag{30}$$

If a is independent of x^2 and x^4, and if a contains x^1 only linearly, then first three PDEs trivially hold, and the last one reduces to $2a_{13} - (a_1)^2 = 0$. We shall see that $a = -\frac{2x^1}{x^3}$ is a solution to the PDE, and therefore the metric

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\dfrac{2x^1}{x^3} & 0 \\ 0 & 1 & 0 & \dfrac{2x^1}{x^3} \end{bmatrix} \tag{31}$$

is Einstein on the coordinate patch $x^3 > 0$ (or $x^3 < 0$). In fact, this is a Ricci flat metric (for the Ricci tensor r_{ij}, see Appendix B).

For such an Einstein metric g, the proper almost complex structure J defined in (3) becomes

$$J\partial_1 = \partial_2, \qquad J\partial_2 = -\partial_1, \qquad J\partial_3 = -\frac{x^1}{x^3}\partial_2 + \partial_4, \qquad J\partial_4 = -\frac{x^1}{x^3}\partial_1 - \partial_3. \quad (32)$$

The condition (9) for J to be integrable in Theorem 3 becomes

$$a_1 - b_1 - 2c_2 = 2a_1 = -\frac{4}{x^3} \neq 0, \qquad a_2 - b_2 + 2c_1 = 2a_2 = 0. \quad (33)$$

Thus, J cannot be integrable.

The proper opposite almost complex structure J' in (22) has the form

$$J'\partial_1 = -\frac{x^1}{x^3}\partial_2 + \partial_4, \qquad J'\partial_2 = -\frac{x^1}{x^3}\partial_1 - \partial_3,$$
$$J'\partial_3 = \left\{1 + \left(\frac{x^1}{x^3}\right)^2\right\}\partial_2 - \frac{x^1}{x^3}\partial_4, \qquad J'\partial_4 = -\left\{1 + \left(\frac{x^1}{x^3}\right)^2\right\}\partial_1 - \frac{x^1}{x^3}\partial_3. \quad (34)$$

Condition (25) for Ω' to be symplectic in Theorem 6 becomes

$$b_2 = 0, \qquad ba_2 - 2a_4 = 0, \qquad a_1 + 2c_2 = a_1 = -\frac{2}{x^3} \neq 0,$$
$$-2aa_1 - 2b_3 + 4c_4 = -\frac{8x^1}{(x^3)^2} \neq 0. \quad (35)$$

Therefore, Ω' is not symplectic.

The J' integrability condition (26) in Theorem 7 becomes

$$a_1 = -\frac{2}{x^3} \neq 0, \qquad c_1 + \tfrac{1}{2}b_2 = 0, \qquad ba_2 - 2a_4 = 0,$$
$$ab_1 - 2b_3 - 4cc_1 - 2bc_2 + 4c_4 = -aa_1 = -\frac{4x^1}{(x^3)^2} \neq 0. \quad (36)$$

Thus J' is not integrable.

Thus, the Walker 4-manifold of this type is not in $\mathcal{K}$ but in $\mathcal{AK}$ (indefinite almost Kähler 4-manifolds) in the 16 subfamilies.

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Appendix A. Curvature tensors R_{ijkl} (nonzero components)

$$R_{1313} = -\tfrac{1}{2}a_{11}, \qquad R_{1314} = -\tfrac{1}{2}c_{11}, \qquad R_{1323} = -\tfrac{1}{2}a_{12}, \qquad R_{1324} = -\tfrac{1}{2}c_{12},$$

$$R_{1334} = \tfrac{1}{2}a_{14} - \tfrac{1}{2}c_{13} - \tfrac{1}{4}a_2b_1 + \tfrac{1}{4}c_1c_2, \qquad R_{1414} = -\tfrac{1}{2}b_{11}, \qquad R_{1423} = -\tfrac{1}{2}c_{12},$$

$$R_{1424} = -\tfrac{1}{2}b_{12}, \qquad R_{1434} = \tfrac{1}{2}c_{14} - \tfrac{1}{2}b_{13} - \tfrac{1}{4}(c_1)^2 + \tfrac{1}{4}a_1b_1 - \tfrac{1}{4}b_1c_2 + \tfrac{1}{4}b_2c_1,$$

$$R_{2323} = -\tfrac{1}{2}a_{22}, \qquad R_{2324} = -\tfrac{1}{2}c_{22},$$

$$R_{2334} = \tfrac{1}{2}a_{24} - \tfrac{1}{2}c_{23} - \tfrac{1}{4}a_1c_2 + \tfrac{1}{4}a_2c_1 - \tfrac{1}{4}a_2b_2 + \tfrac{1}{4}(c_2)^2, \qquad R_{2424} = -\tfrac{1}{2}b_{22},$$

$$R_{2434} = \tfrac{1}{2}c_{24} - \tfrac{1}{2}b_{23} - \tfrac{1}{4}c_1c_2 + \tfrac{1}{4}a_2b_1,$$

$$\begin{aligned} R_{3434} = {} & c_{34} - \tfrac{1}{2}a_{44} - \tfrac{1}{2}b_{33} - \tfrac{1}{4}a(c_1)^2 + \tfrac{1}{4}aa_1b_1 + \tfrac{1}{4}ca_1b_2 - \tfrac{1}{2}cc_1c_2 - \tfrac{1}{2}a_4c_1 \\ & + \tfrac{1}{2}a_1c_4 - \tfrac{1}{4}a_1b_3 + \tfrac{1}{4}ca_2b_1 + \tfrac{1}{4}ba_2b_2 - \tfrac{1}{4}b(c_2)^2 - \tfrac{1}{2}b_3c_2 \\ & + \tfrac{1}{4}a_2b_4 + \tfrac{1}{4}a_3b_1 + \tfrac{1}{2}b_2c_3 - \tfrac{1}{4}a_4b_2. \end{aligned}$$

Appendix B. Ricci tensor r_{ij} (nonzero components)

$$r_{13} = \tfrac{1}{2}a_{11} + \tfrac{1}{2}c_{12}, \qquad r_{14} = \tfrac{1}{2}b_{12} + \tfrac{1}{2}c_{11},$$

$$r_{23} = \tfrac{1}{2}a_{12} + \tfrac{1}{2}c_{22}, \qquad r_{24} = \tfrac{1}{2}b_{22} + \tfrac{1}{2}c_{12},$$

$$r_{33} = \tfrac{1}{2}aa_{11} + ca_{12} + \tfrac{1}{2}ba_{22} - a_{24} + c_{23} - \tfrac{1}{2}a_2c_1 + \tfrac{1}{2}a_1c_2 + \tfrac{1}{2}a_2b_2 - \tfrac{1}{2}(c_2)^2,$$

$$r_{34} = \tfrac{1}{2}ac_{11} + cc_{12} + \tfrac{1}{2}a_{14} - \tfrac{1}{2}c_{13} - \tfrac{1}{2}a_2b_1 + \tfrac{1}{2}c_1c_2 + \tfrac{1}{2}bc_{22} - \tfrac{1}{2}c_{24} + \tfrac{1}{2}b_{23},$$

$$r_{44} = \tfrac{1}{2}ab_{11} + cb_{12} + c_{14} - b_{13} - \tfrac{1}{2}(c_1)^2 + \tfrac{1}{2}a_1b_1 - \tfrac{1}{2}b_1c_2 + \tfrac{1}{2}b_2c_1 + \tfrac{1}{2}bb_{22}.$$

Appendix C. Scalar curvature S

$$S = \sum_{i,j=1}^{4} g^{ij}r_{ij} = a_{11} + 2c_{12} + b_{22}.$$

Appendix D. Einstein tensor $G_{ij} = r_{ij} - \frac{S}{4}g_{ij}$ (nonzero components)

$$G_{13} = \tfrac{1}{4}a_{11} - \tfrac{1}{4}b_{22}, \qquad G_{14} = \tfrac{1}{2}c_{11} + \tfrac{1}{2}b_{12},$$

$$G_{23} = \frac{1}{2}a_{12} + \frac{1}{2}c_{22}, \qquad G_{24} = \frac{1}{4}b_{22} - \frac{1}{4}a_{11},$$

$$\begin{aligned} G_{33} &= \frac{1}{4}aa_{11} + ca_{12} + \frac{1}{2}ba_{22} - a_{24} + c_{23} - \frac{1}{2}a_2c_1 + \frac{1}{2}a_1c_2 \\ &\quad + \frac{1}{2}a_2b_2 - \frac{1}{2}(c_2)^2 - \frac{1}{2}ac_{12} - \frac{1}{4}ab_{22}, \end{aligned}$$

$$\begin{aligned} G_{34} &= \frac{1}{2}ac_{11} + \frac{1}{2}cc_{12} + \frac{1}{2}a_{14} - \frac{1}{2}c_{13} - \frac{1}{2}a_2b_1 + \frac{1}{2}c_1c_2 + \frac{1}{2}bc_{22} \\ &\quad - \frac{1}{2}c_{24} + \frac{1}{2}b_{23} - \frac{1}{4}ca_{11} - \frac{1}{4}cb_{22}, \end{aligned}$$

$$\begin{aligned} G_{44} &= \frac{1}{2}ab_{11} + cb_{12} + c_{14} - b_{13} - \frac{1}{2}(c_1)^2 + \frac{1}{2}a_1b_1 - \frac{1}{2}b_1c_2 + \frac{1}{2}b_2c_1 \\ &\quad + \frac{1}{4}bb_{22} - \frac{1}{4}ba_{11} - \frac{1}{2}bc_{12}. \end{aligned}$$

References

[1] M. Chaichi, E. García-Río, Y. Matsushita, Curvature properties of four-dimensional Walker metrics, Class. Quantum Grav. 22 (2005) 559–577.

[2] S. Haze, private communication.

[3] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry II, Wiley, 1969.

[4] Y. Matsushita, Fields of 2-planes and two kinds of almost complex structures on compact four-dimensional manifolds, Math. Z. 207 (1991) 281–291.

[5] Y. Matsushita, Some remarks on fields of 2-planes on compact smooth 4-manifolds, in: Adv. St. in Pure Math., vol. 22, Kinokuniya, Tokyo, 1993, pp. 153–167.

[6] Y. Matsushita, P. Law, Hitchin-Thorpe-type inequalities for pseudo-Riemannian 4-manifolds of metric signature $(++--)$, Geom. Ded. 87 (2001) 65–89.

[7] Y. Matsushita, Four-dimensional Walker metrics and symplectic structures, J. Geom. Phys. 52 (2004) 89–99.

[8] P. Nurowski, M. Przanowski, A four-dimensional example of a Ricci flat metric admitting almost-Kähler non-Kähler structure, Class. Quantum Grav. 16 (3) (1999) L9–L13.

[9] Y. Petean, Indefinite Kähler–Einstein Metrics on compact complex surfaces, Commun. Math. Phys. 189 (1997) 227–235.

[10] A.G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, Quart. J. Math. Oxford 1 (2) (1950) 69–79.